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LETTER TO THE EDITOR

Orthogonal polynomial expansions for finite group transformations†

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Abstract. A simple tractable method for calculating finite group transformations is given. This is achieved by introducing orthogonal polynomial functions of the generating element of the transformation. The polynomials, which are finite in number because the Cayley-Hamilton theorem applies, depend only on the conjugacy class of the transformation. Orthogonality is defined with respect to the trace operation. Results for SU(2) and the defining representation of SU(3) are examined in some detail.

The objective is to give a simple tractable method for calculating finite group transformations

$$T = \exp(X) \tag{1}$$

where the generator X of the transformation is a Lie algebra representation matrix (Helgason 1978). Since the exponential is defined by an infinite power series, a direct calculation for T is impractical. One feasible technique is to put X in Jordan canonical form, $X = C^{-1}YC$, thereby allowing T to be computed from

$$T = C^{-1}\exp(Y)C. \tag{2}$$

For X diagonalisable, this method requires one to determine both the eigenvalues of X and its eigenvectors.

In this letter, an alternative method is considered which expresses $\exp(X)$ as a polynomial in X of finite degree. The coefficients of the polynomial depend only upon the eigenvalues of X.

Let $M_n(C)$ denote the space of all $(n\times n)$ matrices with complex entries. $M_n(C)$ is an associative algebra of dimension n^2. Fix $X\in M_n(C)$. Let $A(X)$ denote the subalgebra of $M_n(C)$ generated by the unit matrix I and X; $A(X)$ consists of the polynomials in X. However, by the Cayley-Hamilton theorem, X satisfies its own secular equation, a polynomial of degree n. Indeed, X satisfies its minimal polynomial which is of degree $p\leqslant n$. Thus, X^p (and X^q, $q\geqslant p$) may be expressed as a linear combination of powers of X of degree less than p, $X^p = \sum_{r<p} a_r^p X^r$. Therefore, the dimension of $A(X)$ equals p, $A(X) = \mathrm{span}_C\{I, X, X^2, \ldots, X^{p-1}\}$. In particular, $\exp(X)$

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may be expressed as a polynomial of finite degree (cf Rinehart 1955),

$$\exp(\boldsymbol{X}) = \sum_{k=0}^{p-1} b_k \boldsymbol{X}^k. \tag{3}$$

Although one would like a simple explicit formula for the coefficients b_k in this expansion, this is not possible for the monomial basis (3). In order to achieve an explicit result, it is necessary to transform to an orthonormal basis. Thus, consider the inner product

$$\kappa(\boldsymbol{X}, \boldsymbol{Y}) \equiv \frac{1}{n} \operatorname{Tr}(\boldsymbol{X}^+ \boldsymbol{Y}) \qquad \text{for } \boldsymbol{X}, \boldsymbol{Y} \in M_n(\boldsymbol{C}). \tag{4}$$

Note that $\|\boldsymbol{X}\|^2 = \kappa(\boldsymbol{X}, \boldsymbol{X})$ is the Hilbert-Schmidt norm. Moreover, when $\boldsymbol{X}$ and $\boldsymbol{Y}$ are elements of a classical Lie algebra, $\kappa(\boldsymbol{X}, \boldsymbol{Y})$ is proportional to the Killing form. An orthonormal basis for $A(\boldsymbol{X})$ is given by the polynomials $P_k(\boldsymbol{X})$ of degree k (cf Szego 1939),

$$(D_k D_{k-1})^{1/2} P_k(\boldsymbol{X}) \equiv \begin{vmatrix} M_0 & M_1 & M_2 & \dots & M_k \\ M_1 & M_2 & M_3 & \dots & M_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \\ M_{k-1} & M_k & M_{k+1} & \dots & M_{2k-1} \\ \boldsymbol{I} & \boldsymbol{X} & \boldsymbol{X}^2 & \dots & \boldsymbol{X}^k \end{vmatrix} \tag{5}$$

where M_l is the lth moment of $\boldsymbol{X}$ defined by

$$M_l \equiv \frac{1}{n} \operatorname{Tr}(\boldsymbol{X}^l) \tag{6}$$

and D_k is given by the right-hand side of (5) with $\boldsymbol{X}^l$ replaced by M_{k+l}. It can be shown that

$$\kappa[P_k(\boldsymbol{X}), P_l(\boldsymbol{X})] = \delta_{kl} \qquad 0 \leqslant k, l \leqslant p-1 \tag{7}$$

and

$$P_k(\boldsymbol{X}) = 0 \qquad \text{for } k \geqslant P. \tag{8}$$

For example, the latter follows directly from the minimal polynomial result because any column in the determinant (5) with $k \geqslant p$ can be reduced to a combination of those with $k < p$. Specifically, for the qth element of the kth column $\boldsymbol{X}^{k+q} = \boldsymbol{X}^{p+k-p+q} = \Sigma_{r<p} a_r^p \boldsymbol{X}^{r+k-p+q}$ which shows that it is a linear combination of elements of the qth row of columns less than k. As a_r^p is independent of q and the trace of the sum is the sum of the traces, one has the desired result. For a proof of the orthonormality (7), one can adapt, for example, the argument given by Cramér for ordinary polynomial functions (Cramér 1946).

Now it follows immediately that if

$$\exp(\boldsymbol{X}) = \sum_{k=0}^{p-1} c_k P_k(\boldsymbol{X}) \tag{9}$$

then the coefficients are given explicitly by

$$c_k = \kappa[P_k(\boldsymbol{X}), \exp(\boldsymbol{X})] = \frac{1}{n} \operatorname{Tr}[P_k(\boldsymbol{X})^\dagger \exp(\boldsymbol{X})]. \tag{10}$$

The coefficients c_k or b_k only depend upon the conjugacy class of $\boldsymbol{X}$. This is a consequence of the natural algebra isomorphism

$$\begin{aligned} &A(\boldsymbol{X}) \rightarrow A(\boldsymbol{C}\boldsymbol{X}\boldsymbol{C}^{-1}) \\ &P(\boldsymbol{X}) \rightarrow P(\boldsymbol{C}\boldsymbol{X}\boldsymbol{C}^{-1}) = \boldsymbol{C}P(\boldsymbol{X})\boldsymbol{C}^{-1} \end{aligned} \tag{11}$$

for any invertible matrix $\boldsymbol{C}$ and any polynomial $P(\boldsymbol{X}) \in A(\boldsymbol{X})$. Therefore, if $\boldsymbol{Y} = \boldsymbol{C}\boldsymbol{X}\boldsymbol{C}^{-1}$ is a conjugate of $\boldsymbol{X}$,

$$\begin{aligned} \exp(\boldsymbol{Y}) &= \boldsymbol{C} \exp(\boldsymbol{X}) \boldsymbol{C}^{-1} \\ &= \boldsymbol{C} \sum_{k=0}^{p-1} c_k P_k(\boldsymbol{X}) \boldsymbol{C}^{-1} \\ &= \sum_{k=0}^{p-1} c_k P_k(\boldsymbol{Y}). \end{aligned} \tag{12}$$

Since the coefficients are independent of the choice of class representative, we may exploit this freedom and compute the c_k for the simplest possible matrix $\boldsymbol{Y} = \boldsymbol{C}\boldsymbol{X}\boldsymbol{C}^{-1}$. Clearly, if $\boldsymbol{X}$ is diagonalisable, it is most convenient to determine the c_k for $\boldsymbol{Y}$ a diagonal matrix.

As an example, consider the exponentiation of the Pauli spin matrices. Let $\boldsymbol{Y}$ be a two-dimensional diagonal traceless matrix $\boldsymbol{Y} = \mathrm{diag}(\mathrm{i}\theta, -\mathrm{i}\theta)$. Then, the orthogonal polynomials are given by

$$P_0(\boldsymbol{Y}) = \boldsymbol{I} \qquad P_1(\boldsymbol{Y}) = \frac{1}{\mathrm{i}\theta}\boldsymbol{Y} \tag{13}$$

and $c_0 = \cos\theta$, $c_1 = \mathrm{i}(\sin\theta)$. Hence, $\exp(\boldsymbol{Y}) = \cos\theta \boldsymbol{I} + (\sin\theta/\theta)\boldsymbol{Y}$. From this it follows that for a general $\boldsymbol{X} = \boldsymbol{C}^{-1}\boldsymbol{Y}\boldsymbol{C}$, where $\boldsymbol{C} \in \mathrm{SU}(2)$,

$$\exp(\boldsymbol{X}) = \cos\theta \boldsymbol{I} + \frac{\sin\theta}{\theta}\boldsymbol{X} \tag{14}$$

which is, of course, a well known result that can be obtained and written in a variety of ways (e.g. see Helgason 1978, p 149).

An important special case of this method is the exponentiation of an irreducible representation π of a Lie algebra g to generate finite transformations of the corresponding Lie group G,

$$\boldsymbol{T} = \exp[\pi(\boldsymbol{X})]. \tag{15}$$

Consider the adjoint action

$$\boldsymbol{Y} = \mathrm{Ad}_{\mathrm{g}}(\boldsymbol{X}) = \mathrm{g}\boldsymbol{X}\mathrm{g}^{-1} \qquad \text{for } \mathrm{g} \in \mathrm{G}. \tag{16}$$

Since π is a representation,

$$\boldsymbol{T} = \pi(\mathrm{g})^{-1} \exp[\pi(\boldsymbol{Y})]\pi(\mathrm{g}). \tag{17}$$

The adjoint orbit representative $\boldsymbol{Y}$ may be selected to be in a simple form. For the classical groups, $\boldsymbol{Y}$ may be chosen from the Cartan subalgebra h (Burgoyne and

Cushman 1977):

$$
\begin{aligned}
&G = SU(m) && h = \{\mathrm{diag}(i\lambda_1, \ldots, i\lambda_m), \lambda_i \in \boldsymbol{R}\} \\
&G = Sp(m) && h = \{\mathrm{diag}(i\lambda_1, \ldots, i\lambda_m, -i\lambda_1, \ldots, -i\lambda_m), \lambda_i \in \boldsymbol{R}\} \\
&G = O(2m) && h = \left\{\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_m \\ & & & -\lambda_m & 0 \end{pmatrix}, \lambda_i \in \boldsymbol{R}\right\} \\
&G = O(2m+1), && h = \left\{\begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \lambda_m & \\ & & & -\lambda_m & 0 & \\ & & & & & 0 \end{pmatrix}, \lambda_i \in \boldsymbol{R}\right\}.
\end{aligned}
\tag{18}
$$

Then, $\pi(\boldsymbol{Y})$ for $\boldsymbol{Y} \in h$ is given by a sum over the weights of π.

To illustrate this, consider a finite SU(3) transformation (Akyeampong and Rashid 1972 and references therein). A generic element $\boldsymbol{X}$ can be characterised by two numbers, α and β, multiplying the Cartan subalgebra operators $\boldsymbol{Q}_0$ and $\boldsymbol{\Lambda}_0$,

$$\boldsymbol{Y} = g\boldsymbol{X}g^{-1} = \alpha \boldsymbol{Q}_0 + 2\beta \boldsymbol{\Lambda}_0. \tag{19}$$

The eigenvalues of $\boldsymbol{Q}_0$ and $\boldsymbol{\Lambda}_0$ are given for the (λ, μ) irreducible representation by the rule

$$
\begin{aligned}
&\langle \boldsymbol{Q}_0 \rangle = \varepsilon = (2\lambda + \mu) - 3(p+q) = -3\langle \boldsymbol{Y} \rangle \\
&\Lambda = (\mu + p - q)/2 = I \\
&\langle \boldsymbol{\Lambda}_0 \rangle = M_\Lambda = \Lambda - r = \langle \boldsymbol{I}_z \rangle
\end{aligned}
\tag{20}
$$

where $0 \leqslant p \leqslant \lambda$, $0 \leqslant q \leqslant \mu$, $0 \leqslant r \leqslant 2\Lambda$ (Hecht 1965). The (ε, M_Λ) labels are those used in nuclear theory, while $(\boldsymbol{Y}, \boldsymbol{I}_z)$ denote the hypercharge and isospin projection operators of particle physics.

According to the above, one can produce a result for $\exp(\boldsymbol{X})$ by considering $\exp(\boldsymbol{Y})$. Since the eigenvalues of $\boldsymbol{Y}$ are known, $\alpha\varepsilon + 2\beta M_\Lambda$, the $P_k(\boldsymbol{Y})$ can be generated and the constants c_k determined. For example, for the three-dimensional defining space representation of SU(3), $(\lambda, \mu) = (1, 0)$, one has

$$
\begin{aligned}
\exp(\boldsymbol{X}) = {} & [2\alpha(\alpha+\beta)(3\alpha+\beta)\exp(-\alpha+\beta) - 2\alpha(\alpha-\beta)(3\alpha-\beta)\exp(-\alpha-\beta) \\
& + 2\alpha(\alpha+\beta)(\alpha-\beta)\exp(2\alpha)]\boldsymbol{I}/D \\
& + [(3\alpha+\beta)(\alpha-\beta)\exp(-\alpha+\beta) - (3\alpha-\beta)(\alpha+\beta)\exp(-\alpha-\beta) \\
& + 4\alpha\beta\exp(2\alpha)]\boldsymbol{X}/D \\
& + [-(3\alpha+\beta)\exp(-\alpha+\beta) + (3\alpha-\beta)\exp(-\alpha-\beta) + 2\beta\exp(2\alpha)]\boldsymbol{X}^2/D
\end{aligned}
\tag{21}
$$

where $D = 2\alpha(3\alpha-\beta)(3\alpha+\beta)$. For the class of transformations with $\alpha = 0$, which corresponds to rotations in a two-dimensional subspace of the three-dimensional space, this reduces to

$$\exp(\boldsymbol{X}) \xrightarrow[(\beta=\mathrm{i}\beta')]{(\alpha=0)} \boldsymbol{I}+\frac{\sin\beta'}{\beta'}\boldsymbol{X}-\frac{\cos\beta'-1}{(\beta')^2}\boldsymbol{X}^2 \tag{22}$$

which is also an SU(2) spin-1 result (e.g. see Helgason 1978, p 249).

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